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CONTINUOUS SEMILATTICES

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Abstract. Continuous semilattices are semilattices endowed with an order for which the semilattice operation is continuous, i.e., preserves specified joins. For various types of specification of join-continuity, we describe the free continuous semilattices.

Introduction

In this paper, we investigate ordered algebras with one binary operation which is commutative, associative and idempotent (i.e., is a semilattice operation) and which is continuous in the sense of preserving specified joins. The motivation for the study of continuous semilattices, besides providing a natural example of a variety of continuous algebras as studied in [3] stems from computer science: it has been proposed (see for example [9]) that an appropriate semantic model for nondeterminism is that of a semilattice operation. A finite, nondeterministic merging of program schemes, or paths in programs, is to be represented by the binary sum $+$. In addition, to ensure the existence of solutions of recursive equations obtainable as least fixpoints of appropriate functions, the semantic model, which is endowed with a partial order, must have certain specified completeness properties, and the semilattice operation must be continuous. The free continuous semilattice then provides the desired semantic model.

The existence of free continuous semilattices can be proved by a well-known abstract method (that of the Adjoint Function Theorem). However, if these free objects are to be useful, then an explicit description of them is needed.

In this paper, we present such explicit descriptions of a wide variety of free continuous semilattices.

The types of continuity encountered most frequently in computer science are ω-continuity and Δ-continuity. A general approach to the specification of joins, which we follow here, is the concept of a subset system $\mathcal{Z}$, introduced in [5] and defined in Section 1 below.

We study $\boldsymbol{Z}$-continuous semilattices; these are triples $(A, +, \leqslant)$ where $+$ is a semilattice operation on A, and $\leqslant$ is a $\boldsymbol{Z}$-complete order on A such that $+$ is $\boldsymbol{Z}$-continuous.

The free continuous semilattice over an arbitrary set X, for all $\boldsymbol{Z}$, is described in Section 2.

In the third section we investigate the much more difficult problem of describing the free semilattice over a $\boldsymbol{Z}$-complete poset P. For the (trivial) $\boldsymbol{Z}$ consisting of the empty join alone, we prove that this is the convex-set extension $\mathrm{Con}(P)$ of P. Surprisingly, for an $\boldsymbol{\omega}$-complete or $\boldsymbol{\Delta}$-complete poset P of finite width, $\mathrm{Con}(P)$ is also the free $\boldsymbol{\omega}$-continuous semilattice over P. For general P and any $\boldsymbol{Z}$ consisting of directed sets, the free continuous semilattice is described in Section 5 as a suitable completion of $\mathrm{Con}(P)$.

Finally, we remark on the difficulty of generalizing these results to σ-semilattices, or complete semilattices, and provide examples where free continuous infinitary semilattices do not exist.

1. Preliminaries

Definition 1.1. A *subset system* $\boldsymbol{Z}$ (see [5]) is an operator assigning to each poset P a collection $\boldsymbol{Z}(P)$ of subsets of P such that, for each $X \in \boldsymbol{Z}(P)$ and each order-preserving $f: P \to Q$, we have $f(X) \in \boldsymbol{Z}(Q)$. A poset P is *$\boldsymbol{Z}$-complete* iff each $X \in \boldsymbol{Z}(P)$ has a join $\bigvee X$ in P. An order-preserving map $f: P \to Q$ is *$\boldsymbol{Z}$-continuous* iff it preserves all Z-joins, i.e., if $X \in \boldsymbol{Z}(P)$ and $\bigvee X$ exists in P, then $f(\bigvee X) = \bigvee f(X)$.

Throughout this paper we assume that the empty set is a $\boldsymbol{Z}$-set, so that each $\boldsymbol{Z}$-complete poset is *strict*, i.e., has a least element $\perp = \bigvee \emptyset$, and each $\boldsymbol{Z}$-continuous map is strict, i.e., preserves $\perp$.

Examples of such subset systems are:

- $\perp$: $\perp(P) = \{\emptyset\}$ for each P;
- $\boldsymbol{\omega}$: $\boldsymbol{\omega}(P)$ consists of all ω-chains and all finite chains in P;
- $\boldsymbol{\Delta}$: $\boldsymbol{\Delta}(P)$ consists of all directed subsets of P;
- $\mathbf{PC}$ (pairwise compatible): $\mathbf{PC}(P)$ consists of those subsets X of P such that $x, y \in X$ implies there exists a $z \in P$ with $x, y \leqslant z$;
- $\boldsymbol{\lambda}$ (for each ordinal λ): consists of all images $f(\lambda)$ of order-preserving maps $f: \lambda \to P$, plus $\emptyset$;
- $\boldsymbol{B}$ (binary joins): $\boldsymbol{B}(P)$ consists of all subsets of P with at most two elements;
- $\mathcal{P}$: $\mathcal{P}(P)$ consists of all subsets of P;
- $\mathcal{P}_n$ (for each cardinal n): $\mathcal{P}_n(P)$ consists of those $X \subseteq P$ with card $X \leqslant n$.

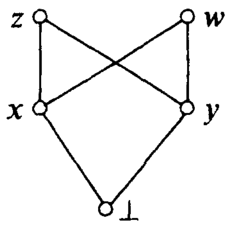
For subset systems $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$, we write $\boldsymbol{Z}_1 \leqslant \boldsymbol{Z}_2$ iff each $\boldsymbol{Z}_2$-complete poset is $\boldsymbol{Z}_1$-complete and each $\boldsymbol{Z}_2$-continuous function is $\boldsymbol{Z}_1$-continuous. For example, $\boldsymbol{\omega} \leqslant \boldsymbol{\Delta} \leqslant \mathbf{PC}$.

Proposition 1.2. (1) *$\boldsymbol{Z} \leqslant \boldsymbol{\Delta}$ iff every $\boldsymbol{Z}$-set is directed.*

(2) $\boldsymbol{B} \leqslant \boldsymbol{Z}$ *iff* $\boldsymbol{Z} \not\leqslant \mathbf{PC}$.

(3) *For each infinite regular ordinal* λ, $\boldsymbol{\lambda} \leqslant \boldsymbol{Z}$ *iff* $\lambda \in \boldsymbol{Z}(\lambda)$.

Proof. (1) If $X \in \boldsymbol{Z}(P)$ is not directed, then let $x, y \in X$ have no common upper bound in X. Define $f: P \to Q$, where

$Q =$ 

as follows:

$$f(p) = \begin{cases} z & \text{if } p \geqslant x \text{ and } p \geqslant y, \\ x & \text{if } p \geqslant x \text{ and } p \not\geqslant y, \\ y & \text{if } p \not\geqslant x \text{ and } p \geqslant y, \\ \perp & \text{else.} \end{cases}$$

Then f is order-preserving and hence, $f(X) \in \boldsymbol{Z}(Q)$. Now, either $f(X) = \{x, y\}$ or $f(X) = \{\perp, x, y\}$ and hence, Q is not $\boldsymbol{Z}$-complete although it is $\boldsymbol{\Delta}$-complete. Hence, $\boldsymbol{Z} \not\leqslant \boldsymbol{\Delta}$. The reverse implication is trivial.

(2) See [1, Proposition 1.4].

(3) If $\boldsymbol{\lambda} \leqslant \boldsymbol{Z}$, then λ is not $\boldsymbol{Z}$-complete, hence, there exists a cofinal subset M of λ with $M \in \boldsymbol{Z}(\lambda)$. Since λ is regular, M is order-isomorphic to λ, and hence, $\lambda \in \boldsymbol{Z}(\lambda)$. Thus $\boldsymbol{\lambda} \leqslant \boldsymbol{Z}$ implies $\lambda \in \boldsymbol{Z}(\lambda)$; the reverse implication is trivial. □

Definition 1.3. For a subset system $\boldsymbol{Z}$, a *separately $\boldsymbol{Z}$-continuous semilattice* (continuous semilattice, for short) is a triple $(A, +, \leqslant)$, where $(A, +)$ is a semilattice, $(A, \leqslant)$ is a $\boldsymbol{Z}$-complete poset, and for each $a \in A$ the map $a + -: A \to A$ preserves the order and nonempty $\boldsymbol{Z}$-joins. That is, $b \leqslant c$ implies $a + b \leqslant a + c$, and for each $X \in \boldsymbol{Z}(A)$ with $X \neq \emptyset$,

$$a + \bigvee X = \bigvee_{x \in X} (a + x).$$

Remark 1.4. The semilattice operation + induces a partial order on A, which, of course, need not be the same as the order $\leqslant$. To avoid confusion, we never refer to the 'semilattice' order, working instead with + as an algebraic operation.

Morphisms between separately $\boldsymbol{Z}$-continuous semilattices are $\boldsymbol{Z}$-continuous maps which are semilattice homomorphisms.

Proposition 1.5. (a) $x \vee y \geqslant x + y$ *whenever* $x \vee y$ *exists, so in particular* $x \geqslant x + \perp$.

(b) *If* $x = x + \perp$ *and* $y = y + \perp$, *then* $x + y = x \vee y$. *Similarly, for any finite subset* $F \subseteq A$ *such that* $x = x + \perp$ *for all* $x \in F$, $\sum F = \bigvee F$.

Proof. (a) Since $x \vee y \geq x$ and $x \vee y \geq y$, we have $x \vee y = (x \vee y) + (x \vee y) \geq x + y$.

(b) We have $x = x + \perp \leq x + y$ and $y = \perp + y \leq x + y$. Given $b \in A$ with $x \leq b$ and $y \leq b$, then $x + y \leq b + b = b$, and hence, $x + y$ is the least upper bound of x and y. □

Remark 1.6. In [3, 5] the notion of a jointly continuous, rather than separately continuous, algebra was discussed.

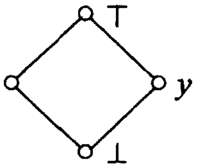
For semilattices, this means that the operation $+ : A \times A \to A$ preserves nonempty Z-joins, i.e.,

$$\bigvee_{(x,y) \in X} (x+y) = \bigvee_{(x,y)\in X} x + \bigvee_{(x,y)\in X} y \quad \text{for all } \emptyset \neq X \in Z(A \times A).$$

Now, if $Z \leq \Delta$, i.e., if each Z-set is directed, then each separately continuous semilattice is jointly continuous, and hence, these two notions coincide. In case $Z \not\leq \Delta$, jointly continuous semilattices are not of interest, as is shown in the following proposition.

Proposition 1.7. *If $Z \not\leq \Delta$, then in any jointly Z-continuous semilattice the order $\leq$ coincides with the semilattice order.*

Proof. For

$P =$ 

either $\{x, y, \perp\}$ or $\{x, y\} \in Z(P)$ (see [11]). For each element a of a continuous semilattice A, we have $(\perp, \perp) \leq (\perp, a)$, $(a, \perp) \leq (a, a)$ in $A \times A$ and hence, either $\{(\perp, \perp), (\perp, a), (a, \perp)\}$ or $\{(\perp, a), (a, \perp)\}$ is a Z-set in $A \times A$. In the first case, the joint continuity of + yields

$$a + a = (\bigvee \{\perp, \perp, a\}) + (\bigvee \{\perp, a, \perp\}) = \bigvee \{\perp + \perp, \perp + a, a + \perp\} = a + \perp$$

and so $a = a + \perp$. In the second case we also obtain $a = a + \perp$, and part (b) of Proposition 1.5 now implies that $a + b = a \vee b$ for all a, b. □

2. Free continuous semilattices over a set

In this section, we give an explicit description, for all Z, of the free continuous semilattice $F(X)$ generated by a set X. This is defined to be a Z-continuous semilattice FX with $X \subseteq FX$, such that every map $f : X \to A$, where A is a continuous semilattice, extends uniquely to a Z-continuous homomorphism $f^* : FX \to A$.

Definition 2.1. The *weight* $w(Z)$ of a subset system Z is the smallest limit ordinal λ with $\lambda \notin Z(\lambda)$, or $\omega(Z) = \infty$ if no such λ exists.

Remark 2.2. The weight is either ∞, or an infinite regular cardinal. For example, $w(\boldsymbol{\omega})=\omega_1$ and $w(\boldsymbol{\Delta})=\infty$.

Theorem 2.3. *For any subset system $\boldsymbol{Z}\leqslant\mathbf{PC}$ the free continuous semilattice $(FX,+,\leqslant)$ over a set X can be described as follows*:

$$FX=\{Y\subseteq X\cup\{\bot\}\mid 0<\text{card } Y<w(\boldsymbol{Z}),\ \textit{and } Y \textit{ infinite}\Rightarrow\bot\in Y\},$$

$$Y+Y'=Y\cup Y',$$

and

$$Y\leqslant Y' \textit{ iff either } Y=Y', \textit{ or } \bot\in Y \textit{ and } Y-\{\bot\}\subseteq Y'.$$

We consider X as a subset of FX by identifying $x\in X$ with $\{x\}\in FX$.

Remark 2.4. For $\boldsymbol{Z}=\boldsymbol{\Delta}$ or $\boldsymbol{\omega}$, and X countable, our FX is precisely Plotkin's 'Power domain' (see [12]). It was observed in [9] that powerdomains are free continuous semilattices.

Proof of Theorem 2.3. (1) FX with $+=\cup$ is clearly a semilattice.

(2) $(FX,\leqslant)$ is $\boldsymbol{Z}$-complete: for each $S\in\boldsymbol{Z}(FX)$, we will show that $\bigvee S\subseteq\bigcup S$, and either S has a largest element, or $\bigvee S=\bigcup S$ and $\bot\in\bigcap S$.

In fact, suppose $S\in\boldsymbol{Z}[FX]$ has no largest element. Then, since $\boldsymbol{Z}\leqslant\mathbf{PC}$, we have $\bot\in Y$ for all $Y\in S$. To prove $\bigvee S=\bigcup S$, it is sufficient to verify card $\bigcup S<w(\boldsymbol{Z})$, for then $\bot\in\bigcup S\in FX$. If $w(\boldsymbol{Z})=\infty$, there is nothing to prove. Assume $n=w(\boldsymbol{Z})$ is a cardinal; we consider n as the set of all ordinals $i<n$. Suppose card $\bigcup S\geqslant n$. Since card $Y<n$ for all $Y\in S$, there clearly exist $Y_i\in S$ for $i<n$ such that $Y_j\nleqslant\bigcup_{k\leqslant i}Y_k$ for $i<j$. We derive a contradiction by proving that $n\in\boldsymbol{Z}(n)$ (which implies $\boldsymbol{n}\leqslant\boldsymbol{Z}$, and this contradicts the definition of weight). Let $h\colon FX\to n$ be the map defined by

$$h(Y)=\bigvee\{i<n\mid Y_i\leqslant Y\}.$$

Since card $Y<n$ and n is regular, the set $\{i<n\mid Y_i\leqslant Y\}$ is bounded and hence, h is well-defined; moreover, h is clearly order-preserving. Since $h(Y_i)=i$, we see that $n=\{h(Y)\mid Y\in S\}\in\boldsymbol{Z}(n)$.

(3) $+$ is order-preserving and continuous. It is obvious that if $Y\leqslant W$, then $U+Y\leqslant U+W$ for U, Y, $W\in FX$.

Next, suppose $\emptyset\neq S\in\boldsymbol{Z}(FX)$ and $U\in FX$; we will show $U+\bigvee S=\bigvee\{U+Y\mid Y\in S\}$. If S has a largest element T, then $U+T$ is the largest element of $\{U+Y\mid Y\in S\}$ because $+$ is order-preserving, and hence, the equality holds. If S does not have a largest element, then, by (2), $\bot\in\bigcap S$ and $\bigvee S=\bigcup S$, thus,

$$\begin{aligned}U+\bigvee S&=U\cup\bigcup S=\bigcup\{U\cup Y\mid Y\in S\}=\bigcup\{U+Y\mid Y\in S\}\\&=\bigvee\{U+Y\mid Y\in S\},\end{aligned}$$

where the last equality comes from the fact that $\bot\in U+Y$ for all $Y\in S$.

(4) Given a continuous semilattice A, we prove that each map $f: X \to A$ has a unique extension to a $\boldsymbol{Z}$-continuous homomorphism $f^*: FX \to A$, defined as follows: for Y finite,

$$f^*(Y) = \begin{cases} \sum f(Y) & \text{if } \perp \notin Y, \\ \perp + \sum \{f(y) \mid \perp \neq y \in Y\} & \text{if } \perp \in Y, \end{cases}$$

and for Y infinite,

$$f^*(Y) = \bigvee_{U \in Q(Y)} f^*(U),$$

where $Q(Y) = \{U \subseteq Y \mid \perp \in U \text{ and } U \text{ is finite}\}$.

(4)(a) We must establish that the join defining $f^*(Y)$, for Y infinite, exists. We are going to prove more, namely that

$$a + f^*(Y) = \bigvee_{U \in Q(Y)} a + f^*(U),$$

for all $a \in A$ and all $Y \in FX$ (finite or infinite) with $\perp \in Y$.

We proceed by induction in card Y.

If Y is finite, the above equality is trivial.

For the induction step, let card $Y = \beta < w(\boldsymbol{Z})$, and assume the equality holds for all sets of smaller cardinality. Let $Y_i \subseteq Y$ $(i < \beta)$ be sets of cardinality $< \beta$ with $Y = \bigcup_{i<\beta} Y_i$ and $\perp \in Y_i \subseteq Y_j$ for $i \leqslant j$. By the induction hypothesis, $\bigvee_{U \in Q(Y_i)} f^*(U) = a_i$ exists for $i < \beta$ and satisfies

$$\bigvee_{U \in Q(Y_i)} (a + f^*(U)) = a + a_i.$$

We thus obtain a β-chain $(a_i)_{i<\beta}$ in A which is a $\boldsymbol{Z}$-set by the definition of weight. Furthermore, for each finite $U \subseteq Y$ with $\perp \in U$, there is $i < \beta$ with $U \subseteq Y_i$ and hence,

$$\bigvee_{i<\beta} a_i = \bigvee_{U \in Q(Y)} f^*(U).$$

Thus,

$$\begin{aligned} a + \bigvee_{U \in Q(Y)} f^*(U) &= a + \bigvee_{i<\beta} a_i = \bigvee_{i<\beta} (a + a_i) \\ &= \bigvee_{i<\beta} \bigvee_{U \in Q(Y_i)} (a + f^*(U)) \quad \text{(by the inductive hypothesis)} \\ &= \bigvee_{U \in Q(Y)} (a + f^*(U)). \end{aligned}$$

(4)(b) $f^*: (FX, \leqslant) \to (A, \leqslant)$ is $\boldsymbol{Z}$-continuous. The fact that f^* is order-preserving follows from its definition and the definition of the order in FX.

Next, let $S \in \boldsymbol{Z}(FX)$; then we must prove $f^*(\bigvee S) \leqslant \bigvee f^*(S)$. We can clearly assume that S does not have a largest element; consequently, $\perp \in \bigcap S$ and $\bigvee S = \bigcup S$ by (2). Then,

$$f^*(\bigvee S) = \bigvee_{U \in Q(\bigcup S)} f^*(U)$$

and it is sufficient to verify $f^*(U) \leq \bigvee_{Y\in S} f^*(Y)$ for any finite $U \subseteq \bigcup S$ with $\perp \in U$. Choose $Y_1, \ldots, Y_n \in S$ with $U = \bigcup_{i=1}^{n} (U \cap Y_i)$. Then, $\perp \in U \cap Y_i$ and hence, $U \cap Y_i \leq Y_i$ for $1 \leq i \leq n$. Then, $f^*(U \cap Y_i) \leq f^*(Y_i) \leq \bigvee_{Y\in S} f^*(Y)$. Thus,

$$f^*(U) = f^*\left(\sum_{i=1}^{n} U \cap Y_i\right) = \sum_{i=1}^{n} f^*(U \cap Y_i) \leq \bigvee_{Y\in S} f^*(Y).$$

(4)(c) $f^* : (FX, +) \to (A, +)$ is a semilattice homomorphism. It is clear that, for finite Y and Y' in FX,

$$f^*(Y) + f^*(Y') = f^*(Y + Y').$$

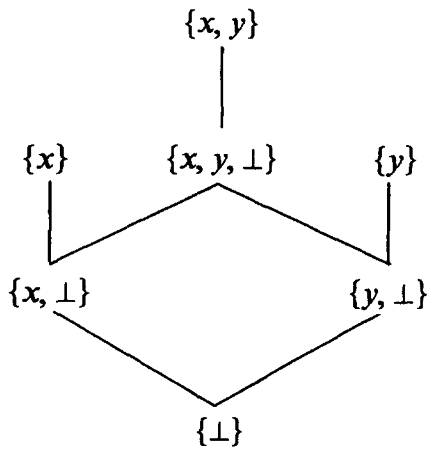
If Y is finite and Y' is infinite, then

$$\begin{aligned} f^*(Y+Y') &= \bigvee_{U\in Q(Y\cup Y')} f^*(U) = \bigvee \{f^*(U) \mid Y \cup \{\perp\} \subseteq U \subseteq Y',\ U \text{ finite}\} \\ &= \bigvee_{U\in Q(Y')} f^*(Y) + f^*(U) \\ &= f^*(Y) + \bigvee_{U\in Q(Y')} f^*(U) \quad \text{(by (4)(a))} \\ &= f^*(Y) + f^*(Y'). \end{aligned}$$

Finally, if Y and Y' are both infinite, then the desired inequality is obtained from the preceding argument, and the fact that $f^*(Y) + f^*(U) = f^*(Y+U)$ for all finite $U \subseteq Y'$ containing $\perp$.

(4)(d) To prove the uniqueness of f^*, first observe that, for $Y \subseteq X$ finite, we have $Y = \sum Y$ if $\perp \notin Y$ and $Y = \perp + \sum (Y - \{\perp\})$ if $\perp \in Y$. The infinite sets $Y \in FX$ are obtained by successive formation of β-joins for $\beta < w(\boldsymbol{Z})$, which are $\boldsymbol{Z}$-joins. $\square$

Example 2.5 (*Free ω-continuous semilattices*). Here, $\boldsymbol{Z} = \boldsymbol{\omega}$ and $w(\boldsymbol{Z}) = \omega_1$. Thus, FX consists of all nonempty finite subsets of $X \cup \{\perp\}$ and those countable subsets which contain $\perp$. For example, $F\{x, y\} = \exp\{x, y, \perp\}$, where $+$ is union, and $\leq$ has the following Hasse-diagram:



Next, we consider subset systems $\boldsymbol{Z} \not\leq \mathbf{PC}$, that is, such that $\boldsymbol{B} \leq \boldsymbol{Z}$ (see Proposition 1.2(2)). The following theorem describes the free continuous semilattice in this case.

Theorem 2.6. *If $\boldsymbol{Z}$ is a subset system with $\boldsymbol{B} \leqslant \boldsymbol{Z}$, then the free continuous semilattice $(F^*X, +, \leqslant)$ over a set X can be described as follows:*

$$F^*X = \{Y \subseteq \mathrm{Fin}(X) \mid \mathrm{card}\ Y < w(\boldsymbol{Z}) \text{ and } y, z \in Y \Rightarrow y \cup z \in Y\},$$

where $\mathrm{Fin}(X)$ *is the set of all nonempty finite subsets of* X,

$$Y + Y' = \{y \cup y' \mid y \in Y, y' \in Y'\},$$

and

$$Y \leqslant Y' \text{ iff } Y \subseteq Y'.$$

We consider X as a subset of F^*X by identifying $x \in X$ with $\{\{x\}\} \in F^*X$.

Proof. (1) $(F^*X, +)$ is obviously a semilattice.

(2) $(F^*X, \leqslant)$ is $\boldsymbol{Z}$-complete. In fact, let $M \in \boldsymbol{Z}(F^*X)$; then we prove that

$$\bigvee M = \bigcup_{1 \leqslant k < \omega} \bigcup_{Y_i \in M} (Y_1 + \cdots + Y_k).$$

We first prove that $\mathrm{card} \bigcup M < w(\boldsymbol{Z})$. Denote, for short, $n = w(\boldsymbol{Z})$ and consider n as the well-ordered set of all smaller ordinals. Assuming $\mathrm{card} \bigcup M \geqslant n$, we can find pairwise distinct $y_j \in \bigcup M$ for $j < n$ and then we can define an order-preserving map $h: F^*X \to n$ by

$$h(Y) = \bigvee_{y_j \in Y} j$$

(the join exists since $\mathrm{card}\ Y < n$). Obviously, $n = h[M] \in \boldsymbol{Z}(n)$, a contradiction to the definition of n. This proves $\mathrm{card} \bigcup M < w(\boldsymbol{Z})$.

Next, the set $T = \{y_1 \cup \cdots \cup y_k \mid 0 < k < \omega, y_1, \ldots, y_k \in \bigcup M\}$ also has cardinality $< n$, and it is closed under finite unions. Hence, $T \in F^*X$ and T is the least set (under $\subseteq$) containing all $Y \in M$ and closed under finite union and thus, $T = \bigvee M$. Clearly,

$$T = \bigcup_{k \geqslant 1} \bigcup_{Y_i \in M} (Y_1 + Y_2 + \cdots + Y_k).$$

(3) $+$ is order-preserving and continuous. The first is clear. The equality

$$Y + \bigvee M = \bigvee_{Y' \in M} (Y + Y') \qquad (M \in \boldsymbol{Z}(F^*X),\ Y \in F^*X)$$

follows from the fact that $Y + \bigvee M$ is the set of all sets $Y + (Y_1 + \cdots + Y_k)$ for $Y_1, \ldots, Y_k \in M$, and $\bigvee (Y + Y')$ is the set of all sets $(Y + Y_1) + \cdots + (Y + Y_k)$ for $Y_1, \ldots, Y_k \in M$.

(4) For each continuous semilattice A and each map $f: X \to A$ we define $f^*: F^*X \to A$ as follows:

$$f^*(Y) = \bigvee_{y \in Y} \sum_{x \in y} f(x).$$

The map f^* is well-defined, because each $y \in Y$ is finite and each Y has cardinality $<n$: since $\boldsymbol{B} \leqslant \boldsymbol{Z}$ and $\boldsymbol{\lambda} \leqslant \boldsymbol{Z}$ for each $\lambda < n$ $(= w(\boldsymbol{Z}))$, it follows that $\mathcal{P}_n \leqslant \boldsymbol{Z}$. Thus A, being $\boldsymbol{Z}$-complete, has joins of all sets of cardinality $<n$. To prove that f^* is additive, we use the following fact: $\bigvee B + \bigvee C = \bigvee_{b \in B} \bigvee_{c \in C} (b+c)$ whenever B, C have cardinalities $<n$. This follows from the $\mathcal{P}_n$-continuity of A:

$$\bigvee B + \bigvee C = \bigvee_{b \in B} \left(b + \bigvee C\right) = \bigvee_{b \in B} \bigvee_{c \in C} (b+c).$$

Thus, given Y, $Y' \in F^*X$ then

$$\begin{aligned} f^*(Y) + f^*(Y') &= \bigvee_{y \in Y} \bigvee_{y' \in Y'} \left(\sum_{x \in y} f(x) + \sum_{x' \in y'} f(x') \right) \\ &= \bigvee_{y \cup y' \in Y + Y'} \sum_{x \in y \cup y'} f(x) \\ &= f^*(Y + Y'). \end{aligned}$$

Finally, f^* is $\boldsymbol{Z}$-continuous because given $M \in \boldsymbol{Z}(F^*X)$,

$$f^*(Y_1 + \cdots + Y_k) = \sum_{i=1}^{k} f^*(Y_i) \leqslant \bigvee_{i=1}^{k} f^*(Y_i) \leqslant \bigvee_{Y \in M} f^*(Y)$$

for all $Y_1, \ldots, Y_k \in M$. Thus,

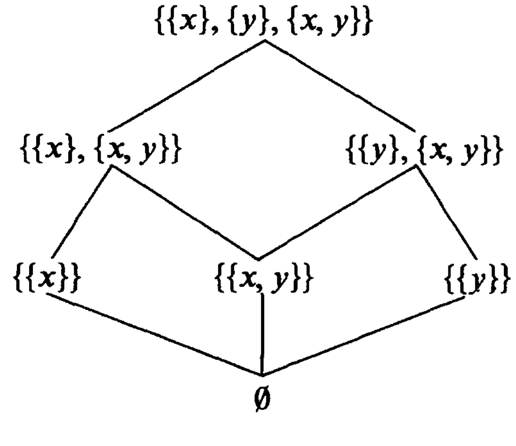
$$f^*(\bigvee M) = \bigvee f^*(Y_1 + \cdots + Y_k) \leqslant \bigvee_{Y \in M} f^*(Y),$$

and the reverse inclusion is clear, because f^* is order-preserving.

The $\boldsymbol{Z}$-continuous homomorphism f^* is unique because

$$Y = \bigvee_{y \in Y} \sum_{x \in y} \{\{x\}\} \quad \text{for each } Y \in F^*X. \qquad \square$$

Example 2.7. Let $\boldsymbol{Z} = \boldsymbol{B}$. For $X = \{x, y\}$ we have $\mathrm{Fin}(X) = \{\{x\}, \{y\}, \{x, y\}\}$ and hence, $(F^*X, \leqslant)$ is the following poset.



3. The convex-set extension of posets

Definition 3.1. In the previous section, we described free continuous semilattices generated by (nonordered) sets, for all subset systems Z. We now turn to the study of the free continuous semilattice over a Z-complete poset P, which is, by definition, a Z-continuous semilattice $(FP, +, \leq)$ such that

(a) P is a subposet of FP, i.e., $x \leq y$ in P iff $x \leq y$ in FP,

(b) the inclusion map $P \to FP$ is Z-continuous, i.e., for all $M \in Z(P)$ the joins $\bigvee M$ in P and in FP coincide,

(c) each Z-continuous map $P \to A$, where A is a continuous semilattice, has a unique extension to a Z-continuous homomorphism $FP \to A$.

In this section, we will describe the free *strictly ordered semilattice*, i.e., $Z = \perp$, over an arbitrary strict poset P, and show that for an ω-complete P of finite width, this semilattice is also the free ω-continuous semilattice over P. A strictly ordered semilattice is just a triple $(A, +, \leq)$ where $(A, +)$ is a semilattice and $\leq$ is a strict order such that each map $a + - : A \to A$ is order-preserving.

Definition 3.2. For a poset P, a subset $M \subseteq P$ is *convex* iff $x \leq y \leq z$ and $x, z \in M$ imply $y \in M$. For a subset $M \subseteq P$, the convex closure $\mathrm{Cl}\, M$ is the smallest convex set containing M.

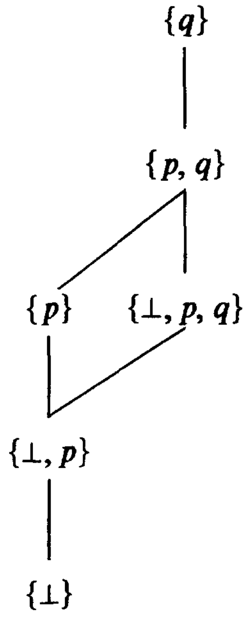
For each strict poset P, we define a poset $\mathrm{Con}(P)$ extending P, which is called the *convex set extension of* P, as follows: the elements of $\mathrm{Con}(P)$ are all finitely generated convex sets, i.e., sets of the form $\mathrm{Cl}\, F$ for nonempty finite $F \subseteq P$. Given $M, N \in \mathrm{Con}(P)$, put $M \leq N$ iff

(i) for each $x \in M$, there is a $y \in N$ with $x \leq y$;

(ii) for each $y \in N$ there is an $x \in M$ with $x \leq y$.

We consider P as a subposet of $\mathrm{Con}(P)$ by identifying $x \in P$ with $\{x\} \in \mathrm{Con}(P)$.

Example 3.3. Let $P = \{\perp, p, q\}$ be the chain $\perp \leq p \leq q$. Then $\mathrm{Con}(P)$ is the following poset:



Remark 3.4. The order $\leqslant$ above is called the Egli–Milner order; see [12, 13].

An alternative description of $\mathrm{Con}(P)$, which is used by Plotkin, is the following. Let Q be the set of all finite nonempty subsets of P, with the Egli–Milner order as described above; here, this is only a preorder, and $\mathrm{Con}(P)$ is the quotient of Q modulo the equivalence relation induced by the preorder.

Yet another description of $\mathrm{Con}(P)$ is as follows: a subset $M \subseteq P$ is *reduced* if it contains no three-element chains. $\mathrm{Con}(P)$ is isomorphic to the poset of all finite reduced sets with the Egli–Milner order. (In fact, with each finitely generated convex set M we can associate the unique reduced set generating M, viz., the set of all minimal and all maximal elements of M. Note that $M \leqslant N$ iff $\mathrm{Cl}\, M \leqslant \mathrm{Cl}\, N$.)

Proposition 3.5. *For any strict poset P, the free strictly ordered semilattice over P is* $(\mathrm{Con}(P), +, \leqslant)$ *where* $M + N = \mathrm{Cl}(M \cup N)$ *and* $\leqslant$ *is the Egli–Milner order.*

Proof. First, $(\mathrm{Con}(P), +)$ is clearly a semilattice, $(\mathrm{Con}(P), \leqslant)$ is a strict poset with least element $\{\perp\}$, and $+$ is order-preserving.

Let $(A, +, \leqslant)$ be a strictly ordered semilattice. For each $\perp$-continuous map $f: P \to A$, we extend f as follows:

$$f^*(\mathrm{Cl}\, F) = \textstyle\sum f(F),$$

for each finite nonempty $F \subseteq P$. Then, $f^*: \mathrm{Con}(P) \to A$ is well-defined, because $\mathrm{Cl}\, F = \mathrm{Cl}\, G$ implies $\sum f(F) = \sum f(G)$. In fact, for each $m \in F$ there exists an $n \in G$ with $m \leqslant n$ and conversely, and hence, $\sum f(F) \leqslant \sum f(G)$, analogously with $\geqslant$. It is easy to verify that f^* is a strict order-preserving semilattice homomorphism. It is unique, because, in $(\mathrm{Con}(P), +)$, $\mathrm{Cl}\, F = \sum F$ for each finite nonempty $F \subseteq P$. $\square$

Below we shall see that $\mathrm{Con}(P)$ need not be $\boldsymbol{Z}$-complete even if P is $\boldsymbol{Z}$-complete. Nevertheless the semilattice operation $+$ is $\boldsymbol{\Delta}$-continuous, as is shown in the following lemma.

Lemma 3.6. *For any strict poset P, if $D \subseteq \mathrm{Con}(P)$ is a nonempty directed set such that $\bigvee D$ exists in $\mathrm{Con}(P)$, then $M + \bigvee D = \bigvee_{N \in D} (M + N)$, for all $M \in \mathrm{Con}(P)$.*

Proof. Since $+$ is order-preserving, we clearly have $M + N \leqslant M + \bigvee D$ for each $N \in D$. Suppose $Q \in \mathrm{Con}(P)$ satisfies $M + N \leqslant Q$ for all $N \in D$; then we will prove that $M + \bigvee D \leqslant Q$.

First, we prove that for each $d \in M + \bigvee D$, there exists a $q \in Q$ with $d \leqslant q$. It is sufficient to prove this for $d \in M \cup \bigvee D$. This is clear if $d \in M$ (since $M + N \leqslant Q$ for any $N \in D$). If $d \in \bigvee D$, then, for any $N \in D$, let Q_N be the set of all maximal elements $q \in Q$ with $a \leqslant q$ for some $a \in N$; then Q_N is a finite convex set and hence, $Q_N \in \mathrm{Con}(P)$. Since $M + N \leqslant Q$, we have $N \leqslant Q_N$. Further, $N \leqslant N'$ implies $Q_N \supseteq Q_{N'}$. Since Q has only finitely many maximal elements and D is directed, there exists an $N \in D$ with $Q_N = Q_{N'}$ for all $N' \geqslant N$. Thus, $N' \leqslant Q_{N'} = Q_N$ for all $N' \geqslant N$, and hence, $D \leqslant Q_N$. Thus, there exists a $q \in Q_N \subseteq Q$ with $d \leqslant q$.

Second, we prove that, for each $q \in Q$, there exists a $d \in M + \bigvee D$ with $d \leq q$. If $q \geq m$ for some $m \in M$, we are finished. If not, then, for each $N \in D$, $M + N \leq Q$ implies that there exists a $d_N \in N$ with $q \geq d_N$ and so $N \leq \{q\} + \bigvee D$ for each $N \in D$. Consequently, $\bigvee D \leq \{q\} + \bigvee D$ and hence, there exists $d \in \bigvee D$ with $q \geq d$, as required. □

Definition 3.7. A poset P has *finite width* iff there exists a natural number k such that all antichains in P have at most k elements.

We denote by **Chain** the subset system with **Chain**(P) the collection of all well-ordered subsets of P. This subset system is equivalent to $\boldsymbol{\Delta}$ (i.e., $\boldsymbol{\Delta} \leq$ **Chain** as well as **Chain** $\leq \boldsymbol{\Delta}$) see [10]. We shall consider subset systems $\boldsymbol{Z} \subseteq$ **Chain** (i.e., such that all $\boldsymbol{Z}$-sets are well-ordered).

Theorem 3.8. *For each subset system* $\boldsymbol{Z} \subseteq$ **Chain** *and each* $\boldsymbol{Z}$*-complete poset* P *of finite width,* $\mathrm{Con}(P)$ *is the free continuous semilattice over* P.

Proof. We work with finite, nonempty sets as elements of $\mathrm{Con}(P)$, see Definition 3.2. Note that for $z_1, \ldots, z_k \in P$, the reduction of $\{z_1, \ldots, z_k\}$ is $\sum_{k=1}^{n} z_k$ in $\mathrm{Con}(P)$.

(1) By a weak chain we mean a collection Z_i ($i < \alpha$, α an ordinal) in $\mathrm{Con}(P)$ such that given $z_j \in Z_j$ and $i < j < \alpha$, there exists a $z_i \in Z_i$ with $z_i \leq z_j$. We first prove that for each weak chain there exists a P-chain $z_i \in Z_i$ ($i < \alpha$).

We proceed by transfinite induction in α. The case $\alpha = 1$ is clear. For $\alpha + 1$, we choose any $z_{\alpha+1} \in Z_{\alpha+1}$ and then we apply the induction hypothesis to the weak chain $Z'_i = \{z \in Z_i \mid z \leq z_{\alpha+1}\}$ ($i < \alpha$). Finally, let β be a limit ordinal such that all $\alpha < \beta$ satisfy the statement. We construct $z_i \in Z_i$ by transfinite induction on i, in such a way that given α with $i < \alpha < \beta$, there is a P-chain $y_j \in Z_j$ ($j < \alpha$) with $z_i = y_i$:

(a) $i = 0$. By the induction hypothesis, there are P-chains $z_{\alpha i} \in Z_i$ ($i < \alpha$) for all $\alpha < \beta$. Since Z_0 is finite, there exists a $z_0 \in Z_0$ with $z_0 = z_{\alpha 0}$ for arbitrarily large α ($< \beta$).

(b) Assuming that $z_i \in Z_i$ are given for all $i < i_0$, we present z_{i_0}. For each $i < i_0$ and α with $i_0 \leq \alpha < \beta$, there exists a P-chain $z_{i\alpha j} \in Z_j$ ($j < \alpha$) with $z_i = z_{i\alpha i}$. Since Z_{i_0} is finite, there exists a $z_{i_0} \in Z_{i_0}$ with $z_{i_0} = z_{i\alpha i_0}$ for arbitrarily large $i < i_0$ and $\alpha < \beta$. Then, $z_i \leq z_{i_0}$ for all $i < i_0$ and z_{i_0} fulfills the induction hypothesis.

(2) For each chain Z_i ($i < \alpha$) in $\mathrm{Con}(P)$, each $j < \alpha$, and each $z \in Z_j$ there exists a P-chain $z_i \in Z_i$ ($i < \alpha$) with $z = z_j$.

We prove this by transfinite induction on j. If $j = 0$, we put $z_0 = z$; given z_i, we choose any $z_{i+1} \in Z_{i+1}$ with $z_i \leq z_{i+1}$ (which exists since $Z_i \leq Z_{i+1}$) and given a limit ordinal i, we choose any $z_i \in Z_i$ with $z_k \leq z_i$ for all $k < i$ (which exists because $Z_k \leq Z_i$ and hence, there exists $t_k \in Z_i$ with $z_k \leq t_k$—since Z_i is finite, there is a $z_i \in Z_i$ with $t_k = z_i$ for arbitrarily large k). For $j > 0$, we find the 'upper' part of the chain (z_i for all $i > j$) by the same argument as for $j = 0$, and the 'lower' part is obtained by applying (1) to the weak chain $Z'_i = \{z \in Z_i \mid z \leq z_j\}$ ($i < j$).

(3) For each chain Z_i $(i<\alpha)$ in $\mathrm{Con}(P)$ there exist finitely many chains z_{mi} $(i<\alpha)$ in P, $m=1,2,\ldots,k$, such that the chains Z_i $(i<\alpha)$ and $\sum_{n=1}^{k} z_{ni}$ $(i<\alpha)$ are mutually cofinal in $\mathrm{Con}(P)$.

In fact, let M be a set of P-chains $z_i\in Z_i$ $(i<\alpha)$ maximal with respect to the property that no two distinct chains of M are mutually cofinal in P. By (1), we have $M\neq\emptyset$. Let us verify that M is finite. If m denotes the width of P, then each (reduced!) set Z_i has at most $2m$ points: each point of Z_i is either maximal or minimal and therefore, Z_i is a union of two antichains. Now, each two distinct chains of M differ on Z_i for all sufficiently large i, and thus, card $M\leqslant 2m$. Thus, we can write M in the form $M=\{(z_{ni})_{i<\alpha}\mid n=1,2,\ldots,k\}$.

(3)(a) For each $i<\alpha$ there exists a $j<\alpha$ with $Z_i\leqslant\sum_{n=1}^{k} z_{nj}$. In fact, by (2), for each $z\in Z_i$ there is a P-chain $z^t\in Z_t$ $(t<\alpha)$ with $z=z^i$. Since M is maximal, this chain is cofinal with some chain of M and hence, there exist an n and a j with $z\leqslant z_{nj}$. By the finiteness of Z_i we can choose j independently of z. Then, for each $z\in Z_i$, there exists an n with $z\leqslant z_{nj}$. Consequently, $Z_i\leqslant\sum_{n=1}^{k} z_{nj}$.

(3)(b) For each $i<\alpha$ there exists $j<\alpha$ with $\sum_{n=1}^{k} z_{nj}\leqslant Z_j$. Assuming the contrary, we have $\sum_{n=1}^{k} z_{ni}\not\leqslant Z_j$ for all $j<\alpha$, and then there clearly exists a $t_j\in Z_j$ with $z_{ni}\not\leqslant t_j$ for all $n=1,\ldots,k$. Then we get a weak chain $Z_j'=\{t\in Z_j\mid z_{ni}\not\leqslant t \text{ for all } n=1,\ldots,k\}$ $(j<\alpha)$. By (1), there exists a P-chain $z_j'\in Z_j$ $(j<\alpha)$. Since $z_{ni}\not\leqslant z_j'$ for all n and j, the chain z_j' $(j<\alpha)$ is not cofinal with any z_{nj} $(j<\alpha)$, in contradiction to the maximality of M.

(4) $\mathrm{Con}(P)$ is a $\boldsymbol{Z}$-complete poset. Let Z_i $(i<\alpha)$ be a $\boldsymbol{Z}$-chain in $\mathrm{Con}(P)$. By (3), we may assume $Z_i=\sum_{n=1}^{k} z_{ni}$ $(i<\alpha)$, where z_{ni} $(i<\alpha)$ is a chain in P for every $n=1,\ldots,k$.

Let us prove that the chain has an upper bound in $\mathrm{Con}(P)$. If not, define order-preserving maps $g_n:\mathrm{Con}(P)\to P$ by

$$g_n(X)=z_{ni}\quad\text{where } i=\bigvee_{Z_j\leqslant X} j.$$

Thus, $z_{ni}=g_n(Z_i)$ $(i<\alpha)$ is a $\boldsymbol{Z}$-chain in P for every $n=1,\ldots,k$. As P is $\boldsymbol{Z}$-complete, $\bigvee_{i<\alpha} z_{ni}$ exists in P and then it is easy to see that $\sum_{n=1}^{k}\bigvee_{i<\alpha} z_{ni}$ is an upper bound for Z_i in $\mathrm{Con}(P)$—a contradiction.

We have proved that the chain Z_i $(i<\alpha)$ has an upper bound T in $\mathrm{Con}(P)$. For each n and i there exists $t_{ni}\in T$ with $t_{ni}\geqslant z_{ni}$, and since T is finite, we can choose this element independently of i: there is $z_{n\alpha}\in T$ with $z_{n\alpha}\geqslant z_{ni}$, for all $i<\alpha$ and $n=1,\ldots,k$. Then, using maps $g_n:\mathrm{Con}(P)\to P$ formally defined as above, we see that z_{ni} $(i<\alpha)$ are $\boldsymbol{Z}$-chains in P and hence, $\bigvee_{i<\alpha} z_{ni}$ exist in P for $n=1,\ldots,k$. By continuity of $+$ in $\mathrm{Con}(P)$ (see Lemma 3.6), we have

$$\bigvee_{i<\alpha} Z_i=\bigvee_{i<\alpha}\sum_{n=1}^{k} z_{ni}=\sum_{n=1}^{k}\bigvee z_{ni}.$$

(5) The inclusion map $P\to\mathrm{Con}(P)$ preserves all existing joins. (This is trivial.)

(6) $\mathrm{Con}(P)$ is the free continuous semilattice. First, $\mathrm{Con}(P)$ is a continuous semilattice: see (4) and Lemma 3.6.

Next, let A be a continuous semilattice and let $f: P \to A$ be a continuous map. By Proposition 3.5, we have a unique strict, order-preserving homomorphism $f^{\#}: \mathrm{Con}(P) \to A$. It remains to prove that $f^{\#}$ is $\boldsymbol{Z}$-continuous. As we have seen in (4), each $\boldsymbol{Z}$-join has the form $\bigvee_{i<\alpha} Z_i = \sum_{n=1}^{k} \bigvee_{i<\alpha} z_{ni}$, where z_{ni} $(i<\alpha)$ are $\boldsymbol{Z}$-sets in P. Moreover, since Z_i $(i<\alpha)$ and $\sum z_{ni}$ $(i<\alpha)$ are cofinal in $\mathrm{Con}(P)$, their images are cofinal in A and hence,

$$
\begin{aligned}
\bigvee_{i<\alpha} f^{\#}(Z_i) &= \bigvee_{i<\alpha} f^{\#}\left(\sum_{n=1}^{k} z_{ni}\right) \\
&= \bigvee_{i<\alpha} \sum_{n=1}^{k} f(z_{ni}) \qquad (f^{\#} \text{ is a homomorphism}) \\
&= \sum_{n=1}^{k} \bigvee_{i<\alpha} f(z_{ni}) \qquad (A \text{ is continuous}) \\
&= \sum_{n=1}^{k} f\left(\bigvee_{i<\alpha} z_{ni}\right) \qquad (f \text{ is continuous}) \\
&= f^{\#}\left(\sum_{n=1}^{k} \bigvee_{i<\alpha} z_{ni}\right) \qquad (f^{\#} \text{ is a homomorphism}) \\
&= f^{\#}\left(\bigvee_{i<\alpha} Z_i\right). \qquad \square
\end{aligned}
$$

Corollary 3.9. *Let $\boldsymbol{Z} = \boldsymbol{\Delta}$. For each $\boldsymbol{\Delta}$-complete poset of finite width,* Con(P) *is the free continuous semilattice over P.*

This corollary follows from the equivalence of $\boldsymbol{\Delta}$ and **Chain**.

Theorem 3.8 can be improved in case of chains of cofinality $>\omega$; here, we need not assume finite width.

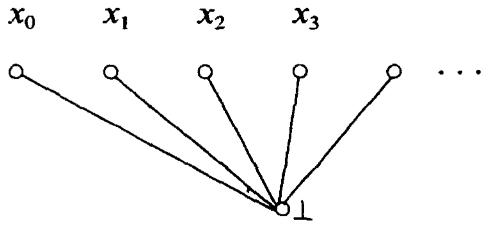
Theorem 3.10. *Let $\boldsymbol{Z} \subseteq$ **Chain** be a subset system with $\omega \notin \boldsymbol{Z}(\omega+1)$. Then, for every $\boldsymbol{Z}$-complete poset P,* Con(P) *is the free continuous semilattice over P.*

Proof. This is analogous to the previous proof. In fact, the only place using the finiteness of the width of P was to show that the set M (in part (3)) is finite. It follows from $\omega \notin \boldsymbol{Z}(\omega+1)$ that cof $\alpha > \omega$. Let there exist a countable subset of M. Then, since each pair of chains of M differs on Z_i for all sufficiently large $i<\alpha$ and cof $\alpha > \omega$, there exists an i_0 such that all these chains differ on Z_{i_0}. This contradicts the finiteness of Z_{i_0}. $\square$

Example 3.11. Let $\boldsymbol{Z} = \boldsymbol{\Delta}$ or $\boldsymbol{Z} = \boldsymbol{\omega}$.

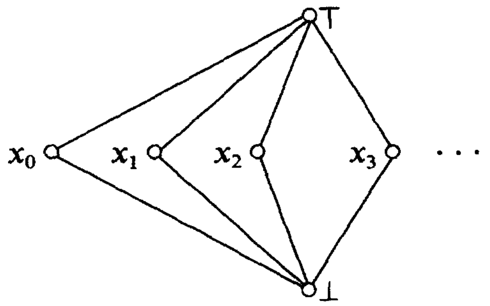
(1) Con(P) need not be $\boldsymbol{Z}$-complete even if P is.

Let P be the countable 'flat' poset



Although P is ω-complete, $\mathrm{Con}(P)$ is not: $\{\perp\}<\{\perp, x_0\}<\{\perp, x_0, x_1\}<\{\perp, x_0, x_1, x_2\}<\cdots$ provides an ω-chain with no join in $\mathrm{Con}(P)$.

(2) We now provide an example of a poset P for which $\mathrm{Con}(P)$ is a continuous semilattice and, nevertheless, $\mathrm{Con}(P)$ is not the free continuous semilattice over P. P is now the poset from Example 3.11(1), with a top element added:



$\mathrm{Con}(P)$ can be depicted as shown in Fig. 1. That is, $\mathrm{Con}(P)=\{P\}\cup M_\perp \cup M_\top \cup M_\emptyset$, where

$M_\perp=\{X\subseteq P \mid X \text{ finite}, \perp\in X, \top\notin X\}$, ordered by set inclusion $\subseteq$,

$M_\top=\{X\subseteq P \mid X \text{ finite}, \perp\notin X, \top\in X\}$, with the reverse order $\supseteq$,

$M_\emptyset=\{X\subseteq P \mid X \text{ finite}, X\neq\emptyset, \perp\notin X, \top\notin X\}$, discretely ordered.

The rest of the order is as indicated in Fig. 1.

It is easy to check that every subset of $\mathrm{Con}(P)$ has a join and hence, $\mathrm{Con}(P)$ is a continuous semilattice, by Lemma 3.6.

However, $\mathrm{Con}(P)$ is not the free continuous semilattice over P. To see this, let $\mathrm{Con}^*(P)$ consist of *all* convex subsets of P, with both the order and semilattice operation defined as they are in $\mathrm{Con}(P)$. The diagram of $\mathrm{Con}^*(P)$ is similar to that of $\mathrm{Con}(P)$; $\mathrm{Con}^*(P)=\{P\}\cup M_\perp^* \cup M_\top^* \cup M_\emptyset^*$, where $M_\perp^*$ consists of all subsets of P which contain $\perp$ but not $\top$, etc. It is easy to prove that $\mathrm{Con}^*(P)$ is a continuous semilattice and that $\mathrm{Con}(P)$ is its subsemilattice, but joins are different; for example, the ω-chain

$$\{x_0, \perp\}\leqslant\{x_0, x_1, \perp\}\leqslant\{x_0, x_1, x_2, \perp\}\leqslant\cdots$$

has join P in $\mathrm{Con}(P)$, and has join $P-\{\top\}$ in $\mathrm{Con}^*(P)$. Consequently, there is no way to extend the identity embedding $P\to\mathrm{Con}^*(P)$ to an ω-continuous semilattice homomorphism $\mathrm{Con}(P)\to\mathrm{Con}^*(P)$.

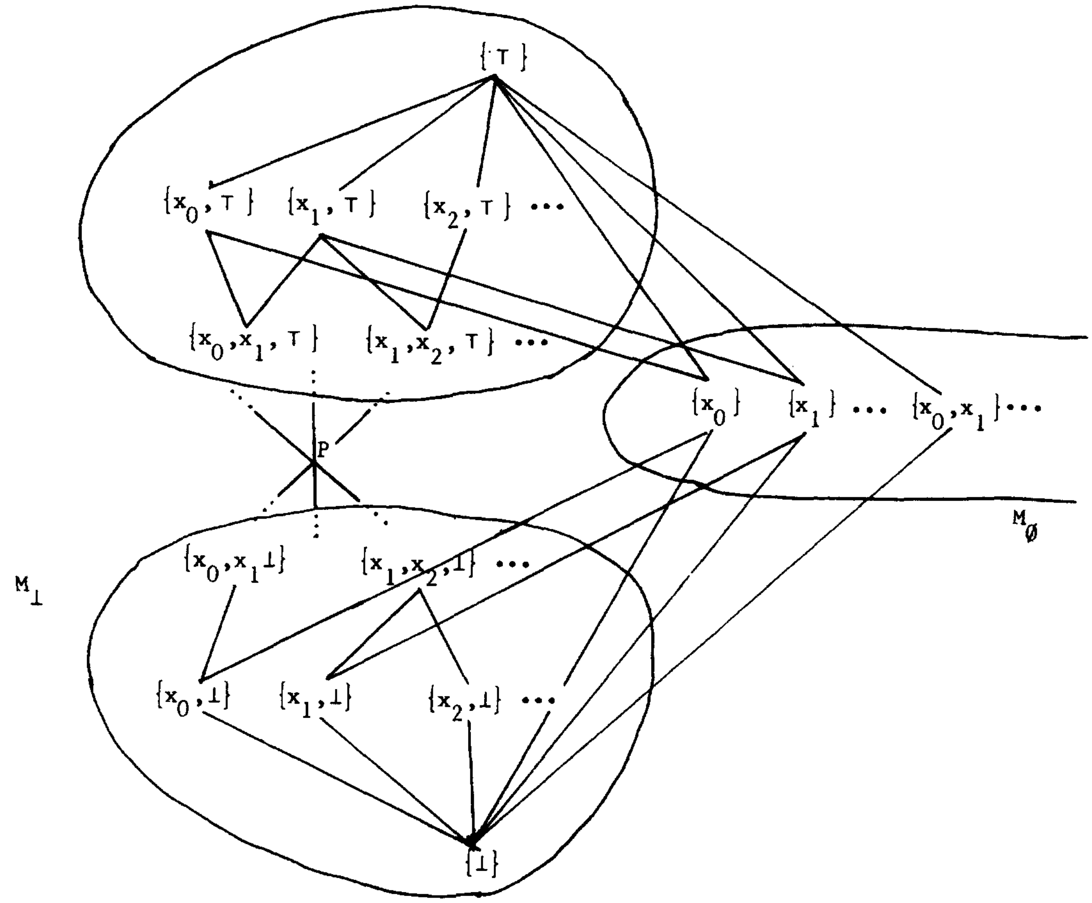


Fig. 1.

4. Completions of ordered algebras

A general result on continuous algebras will be proved here and applied in the next section to continuous semilattices.

Recall that a (strictly) ordered algebra of type Σ is a (strict) poset A equipped with order-preserving operations $\sigma: A^n \to A$ for all $\sigma \in \Sigma$ of arity n (where A^n is ordered componentwise).

Definition 4.1. We are going to use a 'localization' of the concept of continuity. Instead of a subset system, let us consider a collection $\mathcal{M}$ of subsets of a poset P. A map $f: P \to Q$ into a poset Q is said to be *$\mathcal{M}$-continuous* if it preserves order and all existing joins $\bigvee M$ for $M \in \mathcal{M}$.

Generalizing various results in [6, 8] we introduce the free Z-completion preserving $\mathcal{M}$-joins.

Observation 4.2. For any poset P, any collection $\mathcal{M} \subseteq \mathcal{P}(P)$, and any subset system Z, there exists a Z-complete poset $P^{\#}$ such that

(1) P is a subposet of $P^{\#}$ and the inclusion map $P \to P^{\#}$ is $\mathcal{M}$-continuous;

(2) each strict $\mathcal{M}$-continuous map $f: P \to Q$ into a Z-complete poset Q has a unique Z-continuous extension $f^{\#}: P^{\#} \to Q$.

A simple proof of the existence of $P^\#$ can be obtained by applying the Adjoint Functor Theorem, and using the MacNeille completion to prove that the front adjunction $P \to P^\#$ is actually an embedding. It is also possible to describe $P^\#$ as follows. Let $I_{\mathcal{M}}(P)$ be the (complete) poset of all nonempty $\mathcal{M}$-closed ideals (i.e., downward-closed sets $J \subseteq P$ such that $\bigvee M \in J$ for each $M \in \mathcal{M}$ with $M \subseteq J$) ordered by inclusion. Then, $P^\#$ is the $\mathbf{Z}$-join-closure of P in $I_{\mathcal{M}}(P)$, i.e., $P^\#$ is the smallest subposet Q of $I_{\mathcal{M}}(P)$ which contains all principal ideals (identified with elements of P) and such that $\bigvee X \in Q$ for each $X \in \mathbf{Z}(Q)$.

Remark 4.3. (1) P is $\mathbf{Z}$-dense in $P^\#$, i.e., no proper subposet of $P^\#$ containing P is closed under $\mathbf{Z}$-joins.

(2) By choosing $M = \emptyset$, we obtain the concept of the *absolutely free $\mathbf{Z}$-completion*, i.e., the $\mathbf{Z}$-complete poset $P^\#$ such that any strict order-preserving map $P \to Q$ with Q $\mathbf{Z}$-complete has a unique $\mathbf{Z}$-continuous extension to $P^\#$.

Definition 4.4. Let A be a strictly ordered algebra and let $\mathcal{M}$ be a collection of nonempty subsets of A, each having a join in A. Then A is said to be (*separately*) $\mathcal{M}$-continuous provided that each of the maps

$$\sigma(a_1, a_2, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n): A \to A$$

(for $\sigma \in \Sigma$ and $a_j \in A$) is $\mathcal{M}$-continuous and preserves $\mathcal{M}$-sets (i.e., the image of a set in $\mathcal{M}$ is also in $\mathcal{M}$).

Note that for each subset system $\mathbf{Z}$, by choosing $\mathcal{M} = \mathbf{Z}(A) - \{\emptyset\}$, all nonempty $\mathbf{Z}$-sets, we precisely obtain the concept of a separately $\mathbf{Z}$-continuous algebra.

Theorem 4.5. *For each $\mathcal{M}$-continuous algebra A and each subset system $\mathbf{Z}$ there is a unique extension of the operations of A to the free $\mathcal{M}$-continuous $\mathbf{Z}$-completion $A^\#$ such that $A^\#$ is a separately $\mathbf{Z}$-continuous algebra.*

$A^\#$ has the following universal property: each strict $\mathcal{M}$-continuous homomorphism $f: A \to B$ into a separately $\mathbf{Z}$-continuous algebra B has a unique extension to a $\mathbf{Z}$-continuous homomorphism $f^\#: A^\# \to B$.

Remark 4.6. The proof of the extension of the operations is completely analogous to that in [11, Proposition 4].

Proof of Theorem 4.5. (1) We first observe that any (not necessarily strict) $\mathcal{M}$-continuous map $f: A \to B$ with B $\mathbf{Z}$-complete has a unique extension $f^\#: A^\# \to B$ preserving nonempty $\mathbf{Z}$-joins. In fact, we can restrict f to the $\mathbf{Z}$-complete subposet $B_0 = \{b \in B \mid f(\perp) \leq b\}$ and apply the universal property of Observation 4.2(2) to the map $A \to B_0$.

(2) We show that each $\mathcal{M}$-continuous operation $\sigma: A^n \to A$ has a unique extension $\sigma^\#: (A^\#)^n \to A^\#$ separately preserving nonempty $\mathbf{Z}$-joins. In case $n = 1$, this follows

from (1). Suppose $n=2$. For each $a \in A$ the map $\sigma(a, -): A \to A^{\#}$ is $\mathcal{M}$-continuous and hence, it has a unique extension $\sigma^{\#}(a, -): A^{\#} \to A^{\#}$ preserving nonempty $\boldsymbol{Z}$-joins. Further, denoting by $\mathrm{Hom}(A^{\#}, A^{\#})$ the poset of all self-maps of $A^{\#}$ preserving nonempty $\boldsymbol{Z}$-joins ordered point-wise, we then obtain a map $f: A \to \mathrm{Hom}(A^{\#}, A^{\#})$ by $f(a) = \sigma^{\#}(a, -)$. We are going to prove that f is $\mathcal{M}$-continuous. First, $a \leqslant b$ implies $f(a) \leqslant f(b)$ because the set $\{x \in A^{\#} \mid \sigma^{\#}(a, x) \leqslant \sigma^{\#}(b, x)\}$ contains all of A and is clearly closed under $\boldsymbol{Z}$-joins in $A^{\#}$, and hence, the set is all of $A^{\#}$. Analogously, $f(\bigvee M) = \bigvee f(M)$ for each $M \in \mathcal{M}$, because the set $\{x \in A^{\#} \mid \sigma^{\#}(\bigvee M, x) = \bigvee_{m \in M} \sigma^{\#}(m, x)\}$ contains all of A (since σ is $\mathcal{M}$-continuous) and is closed under $\boldsymbol{Z}$-joins (because $\perp \in A$ and each $\sigma^{\#}(a, -)$ preserves nonempty $\boldsymbol{Z}$-joins) and hence, is all of $A^{\#}$. Since $\mathrm{Hom}(A^{\#}, A^{\#})$ is clearly $\boldsymbol{Z}$-complete, there is a unique extension of f to a map $f^{\#}: A^{\#} \to \mathrm{Hom}(A^{\#}, A^{\#})$ preserving joins of nonempty $\boldsymbol{Z}$-sets. This defines the desired operation $\sigma^{\#}$,

$$\sigma^{\#}(a, b) = (f^{\#}(a))(b).$$

For $n > 2$, we proceed analogously.

(3) To prove that $A^{\#}$ has the stated universal property, it suffices to prove that the unique $\boldsymbol{Z}$-continuous map $f^{\#}: A^{\#} \to B$ extending f, whose existence is given by Observation 4.2, is a homomorphism. We again take the case of a binary operation σ; the proof for n-ary operations is analogous.

For $a \in A^{\#}$, define

$$X_a = \{y \in A^{\#} \mid f^{\#}(\sigma(y, a)) = \sigma(f^{\#}(y), f^{\#}(a))\}.$$

If $a \in A$, then X_a contains A and is closed under $\boldsymbol{Z}$-joins by the $\boldsymbol{Z}$-continuity of $f^{\#}$ and the separate continuity of σ in B, and hence, $X_a = A^{\#}$. Now,

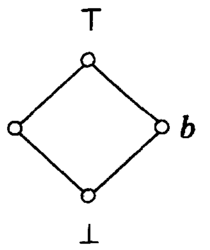
$$X = \{a \in A^{\#} \mid X_a = A^{\#}\}$$

contains A, and is closed under $\boldsymbol{Z}$-joins by the same argument as above, and hence, equals $A^{\#}$. This establishes the fact that $f^{\#}$ preserves σ. $\square$

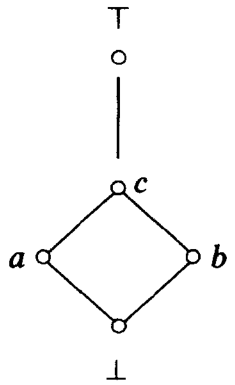
Proposition 4.7. *For each subset system $\boldsymbol{Z} \leqslant \boldsymbol{\Delta}$ and each $\mathcal{M}$-continuous semilattice A, the algebra $A^{\#}$ of Theorem 4.5 is a $\boldsymbol{Z}$-continuous semilattice. Conversely, if $\boldsymbol{Z} \not\leqslant \boldsymbol{\Delta}$, then there exists an $\mathcal{M}$-continuous semilattice A such that $A^{\#}$ is not even idempotent.*

Proof. The first statement is a special case of [11, Proposition 5] which states that $A^{\#}$ satisfies the same equations as A does.

Assuming $\boldsymbol{Z} \not\leqslant \boldsymbol{\Delta}$, the poset

$A =$ 

clearly has the property that either $\{a, b\}$ or $\{a, b, \perp\}$ is a $\mathbf{Z}$-set. We consider A as a (continuous) semilattice with $x+y=x\vee y$, and we put $\mathcal{M}=\emptyset$. Then $A^{\#}$ is the following poset



We shall prove that the semilattice operation $+$ on $A^{\#}$ is not idempotent, by proving $c+c=\top$. In fact, if $\{a, b\}\in \mathbf{Z}(A)\subseteq \mathbf{Z}(A^{\#})$, then

$$c+c=(a\vee b)+(a\vee b)=(a+a)\vee(a+b)\vee(b+a)\vee(b+b)=\top$$

and analogously with $\{a, b, \perp\}\in \mathbf{Z}(A^{\#})$. $\square$

5. Free continuous semilattices over $\mathbf{Z}$-complete posets

In Example 3.11(2) we saw an example of a complete poset P such that $\mathrm{Con}(P)$ is a continuous semilattice (for $\mathbf{Z}=\mathbf{\Delta}$ or $\mathbf{Z}=\boldsymbol{\omega}$), but not the free continuous semilattice over P. The reason is that some $\boldsymbol{\omega}$-joins in $\mathrm{Con}(P)$ are not 'free enough'. For each ω-chain $b_0\leq b_1\leq b_2\leq\cdots$ in P, we have ω-chains in $\mathrm{Con}(P)$,

$$\{b_0\}+A\leq\{b_1\}+A\leq\{b_2\}+A\leq\cdots \quad \text{for } A\in\mathrm{Con}(P),$$

the joins of which in $\mathrm{Con}(P)$ turn out to be formed as in the free semilattice. These chains will be called *tight.* All other joins will have to be destroyed. That is, the free continuous semilattice over P turns out to be the free $\mathcal{M}$-continuous $\boldsymbol{\omega}$-completion of $\mathrm{Con}(P)$ (in the sense of Theorem 4.5) for $\mathcal{M}$ equalling all tight ω-chains.

We shall formulate and prove this for any $\mathbf{Z}\subseteq\mathbf{\Delta}$.

Definition 5.1. A *tight Z-set* in $\mathrm{Con}(P)$ is a set of the form

$$\{\{b\}+A \mid b\in B\}$$

for $A\in\mathrm{Con}(P)$ and $B\in \mathbf{Z}(P)$, $B\neq\emptyset$. The collection of all tight $\mathbf{Z}$-sets is denoted by $\mathcal{M}_{\text{tight}}$.

Theorem 5.2. *If $Z \subseteq \Delta$, then the free continuous semilattice over a Z-complete poset P is the free $\mathcal{M}_{\text{tight}}$-continuous Z-completion of* Con(P).

Proof. Con(P) is $\mathcal{M}_{\text{tight}}$-continuous by Lemma 3.6, and the inclusion of P clearly preserves all existing joins.

Moreover, given a Z-continuous map $f: P \to B$ where B is a Z-continuous semilattice, the unique strict order-preserving homomorphism $f^*: \text{Con}(P) \to B$ extending f is $\mathcal{M}$-continuous: for $X \in Z(P)$ and $A \in \text{Con}(P)$, we have

$$\begin{aligned} f^*\Big(\dot{\bigvee_{C\in X}} (A+C)\Big) &= f^*(A + \bigvee X) = f^*(A) + f(\bigvee X) \\ &= f^*(A) + \bigvee_{C\in X} f^*(C) = \bigvee_{C\in X} (f^*(A) + f(C)) \\ &= \bigvee_{C\in X} f^*(A+C). \end{aligned}$$

By Theorem 4.5, there is a unique Z-continuous homomorphism $f^{\#}: (\text{Con}(P))^{\#} \to B$ extending f^*, and hence, f. The uniqueness of the passages $f \rightsquigarrow f^*$ and $f^* \rightsquigarrow f^{\#}$ makes $f^{\#}$ the unique extension of f to a Z-continuous homomorphism. □

Remark 5.3. The description of free continuous semilattices for $Z \subseteq \Delta$ is particularly simple for posets P which are absolutely free Z-completions (see Remark 4.3). If $P = P_0^{\#}$ is an absolutely free Z-completion of the strict subposet P_0, then $\text{Con}(P_0)^{\#}$ (the absolutely free completion of Con(P_0)) is the free continuous semilattice over P. This is proven as follows.

Given a continuous semilattice A, then Z-continuous maps $P \to A$ are just extensions of order-preserving maps $P_0 \to A$ which have unique extensions to strict order-preserving homomorphisms $\text{Con}(P_0) \to A$ and these, in turn, have a unique extension to Z-continuous homomorphisms $\text{Con}(P_0)^{\#} \to A$.

Remark 5.4. In case $Z = \omega$, each ω-algebraic poset P (see [12]) is the absolutely free completion of its subposet B_p of compact elements. The free continuous semilattice $\text{Con}(B_p)^{\#}$ has been described in [12], where it was called a 'powerdomain'.

6. Concluding remarks

From the point of view of category theory, we presented, in Section 2, a description of the left adjoint to the forgetful functor from Z-Sem, the category of Z-continuous semilattices, to Set. Further, for $Z \leq \Delta$, we described the left adjoint to the forgetful functor from Z-Sem to Z-Pos, the category of Z-complete posets (and Z-continuous maps). There is an 'intermediate' forgetful functor: to the category $\text{Pos}_\perp$ of strict

›osets (and strict order-preserving maps). This can be described as follows: *if* $\boldsymbol{Z} \subseteq \boldsymbol{\Delta}$, *hen the free continuous semilattice over a strict poset P is the absolutely free Z-ompletion of* Con(P). The proof is analogous to that of Theorem 4.5 (with $\mathcal{M} = \emptyset$).

An extension of the preceding results to the case of infinitary semilattices, paricularly σ-semilattices (all countable sums) would be desirable, especially in view ›f the recent interest in countable nondeterminism [13]. However, here the situation s much more difficult. Even if free continuous infinitary semilattices exist, it seems hey will be difficult to describe. More important, they need not exist: it is pointed ›ut in the introduction of [7] that there are arbitrarily large separately $\mathcal{P}$-continuous σ-semilattices generated by a countable set, and hence, *there is no free separately* $\mathcal{P}$*-continuous* σ*-semilattice over an infinite set.* We do not know whether there are ree separately $\boldsymbol{\Delta}$-continuous σ-semilattices over an infinite set.

Further, it follows from [7, Theorem 1] that there are arbitrarily large separately $\boldsymbol{B}$-continuous complete semilattices (i.e., having *all* sums) generated by a countable et, and hence there is no free separately *$\boldsymbol{B}$-continuous complete semilattice over an nfinite set.*

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